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Combinatorial Properties, Invariants and Structures of the Action of $S_n \times A_n$ on $X \times Y$

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Abstract

The transitivity, primitivity, rank and subdegrees, as well as pairing of the suborbits associated with the action of the actions of the direct product $S_n \times A_n$, of the symmetric group S_n by the alternating group A_n alternating on the Cartesian product $X \times Y$, where $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_n\}$ are disjoint sets each containing n elements is an area that has never received attention from researchers for a very long time. In this paper, we prove that the action is both transitive and imprimitive when $n \geq 3$. Also, we establish that that the rank is 6 if $n = 3$, but is 4 for all $n \geq 3$. In addition, we show in this paper that the subdegrees associated with the action are $1, (n-1), (n-1), (n-1)^2$. Lastly, we show that all the suborbits corresponding to the action, are self-paired when $n \geq 4$.

Keywords: Direct Product; Symmetric Group; Alternating Group; Action; Rank; Subdegrees; Suborbital.

1. Notation and preliminary results

Definition 1.1. Let G be a group and X a non-empty set. Then G acts on the left of X if there exists a function $G \times X \to X$ such that $(g_1 g_2)x = g_1(g_2)x$ and $ex = x$ where e is the identity in $G, x \in X$ and $g_1, g_2 \in G$. The action of G on the right of X can be defined in a similar way. In this case, X is called a G-set.

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Definition 1.2. Suppose a group G act on a set X. Define a relation $x \sim y$ on X iff there exist a $g \in G$ such that $y = gx$. This defines an equivalence relation on the set X.

The equivalence class containing x is called the orbit of x which is $Orb_G(x) = \{g_x | g \in G\}$. Since any set is a disjoint union of equivalence classes under an equivalence relation, it follows that if G acts on X then X is a union of disjoint orbits.

Theorem 1.1. Let G be a finite group acting on a set X. The number of orbits of G is $\frac{1}{|G|}\sum_{g\in G}|fix(g)|$ where $fix(g) = \{x \in X | gx = x\}$.

Theorem 2.3 is called the Cauchy-Frobenius Lemma [3]

Definition 1.4. Let G act on a set X and let $x \in X$. The stabilizer in G of x denoted by $Stab_G(x)$ is the subset $Stab_G(x) = \{g \in G | gx = x\}$. In this case $Stab_G(x)$ forms a subgroup of G called the isotropy group of x. It is also denoted by G_x.

Theorem 1.2. Let G be a group acting on a finite set X and $x \in X$. Then $|Orb_G(x)| = |G: Stab_G(x)|$

Theorem 1.2 is called the Orbit-Stabilizer Theorem [3]

Definition 1.5. The action of a group G on the set X is said to be transitive if for each pair of points $x, y \in X$, there exists $g \in G$ such that $gx = y$; in other words, if the action has only one orbit, $Orb_G(x) = X$.

Definition 1.6. Let G act transitively on a set X and let Y be a subset of X such that $|Y|$ is a factor of $|X|$. Then if $gY = Y$ or $gY \cap Y = \emptyset$ for all $g \in G$, then Y is called a block of the action. Clearly $\emptyset$, the set X and the singleton subsets of X form blocks, called the trivial blocks. If these are the only blocks, then G is said to act primitively on X; otherwise G acts imprimitively.

Definition 1.7. Suppose G is a group acting transitively on a set X and let G_x be the stabilizer in G of a point $x \in X$. The orbits $\Delta_0 = \{x\}, \Delta_1, \Delta_2, \ldots, \Delta_{(r-1)}$ of G_x on X are known as suborbits of G. In this case r is called the rank of G while the sizes $n_i = |\Delta_i| (i = 0,1,\ldots,r-1)$, often called the lengths of the suborbits, are known as the subdegrees of G. It can be shown that both r and the cardinality of the suborbits Δ_i $(i = 0,1,\ldots,r-1)$ are independent of the choices of $x \in X$

Definition 1.8. Let G be a group acting transitively on a set X and let Δ be an orbit of G_x on X. Define

$\Delta^* = \{gx | g \in G, x \in g\Delta\}$. Then Δ^* is also an orbit of G_x and is called the G_x-orbit (or G-suborbit) paired with Δ [2]. Clearly, $|\Delta| = |\Delta^*|$ and $\Delta^{**} = \Delta$. If $\Delta^* = \Delta$, then Δ is said to be self-paired.

Definition 1.9. Suppose G acts on X. Then G acts on $X \times X$ also by $g(x, y) = (gx, gy), g \in G, x, y \in X$. If $O \subseteq X \times X$ is a G-orbit, then for a fixed $x \in X, \Delta = \{y \in X | (x, y) \in O\}$ is a G_x-orbit. Conversely, if $\Delta \subseteq X$ is a G_x-orbit, then $O = \{(gx, gy) | g \in G, y \in \Delta\}$ is a G-orbit on $X \times X$. In this case Δ is said to correspond to O.

The G-orbits on $X \times X$ are called suborbitals.

Definition 1.10. Let $O_i \subseteq X \times X$, $(i = 0,1,2,\ldots,r-1)$ be a suborbital. A suborbital graph Γ_i is formed by taking X as the points of Γ_i and including a directed line from x to y $(x, y \in X)$ if and only if $(x, y) \in O_i$. Thus each suborbital O_i determines a suborbital graph Γ_i. Now $O_i^* = \{(x, y) | (y, x) \in O_i\}$ is also a G-orbit.

Definition 1.11. Let G be transitive on X and let Γ be the suborbital graph corresponding to the suborbit Δ. Then Γ is undirected if Δ is self-paired and directed otherwise [1].

2. Transitivity and primitivity of the action of $G = S_n \times A_n$ on $X \times Y$

Theorem 2.1. The action of $S_n \times A_n$ on $X \times Y$ is transitive if and only if $n \geq 3$.

Proof. Consider the action of a group $G = S_2 \times A_2$ on the set $X \times Y$ where $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$ so that $X \times Y = \{(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2), (x_2, y_3)\}$. In this case $S_2 \times A_2 = \{(e_X, e_Y), ((x_1 x_2), e_Y)\}$ where e_X is the identity element in S_2 and e_Y is the identity in A_2. Clearly, $H = Stab_G(x_1, y_1) = \{(e_X, e_Y)\}$ and by Theorem 1.2

$$|Orb_G(x_1, y_1)| = |G\colon H|$$

$$= \frac{|G|}{|H|}$$

$$= \frac{2}{1}$$

$$\neq |\, X \times Y\, |$$

Therefore, the action is intransitive for $n = 2$.

Now, let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ for $n \geq 3$. In this case $|G| = \frac{n!n!}{2}$ and $|X \times Y| = n^2$. Suppose $H = Stab_G(x_1, y_1) = \{(g, g') \in S_n \times A_n | g x_1 = x_1, g' y_1 = y_1\}$. Clearly, $g \in S_n$ fixes $x_1 \in X$ if and only if x_1 belongs to a 1-cycle of g so that $\{g \in S_n | g x_1 = x_1\} \cong S_{n-1}$. Also, $\{g' \in A_n | g' y_1 = y_1\} \cong A_{n-1}$.

Thus, $H \cong S_{n-1} \times A_{n-1}$ and it follows that $H = \frac{(n-1)!(n-1)!}{2}$. Now, by Theorem 1.2,

$$|Orb_G(x_1, y_1)| = |G\colon H|$$

$$= \frac{|G|}{|H|}$$

$$= \frac{\frac{n!\,n!}{2}}{\frac{(n-1)!\,(n-1)!}{2}}$$

$$= n^2$$

$$= |X \times Y|$$

Therefore, the action is transitive for all $n \geq 3$.

Theorem 2.2. The action of $G = S_n \times A_n$ on $X \times Y$ is imprimitive for $n \geq 3$.

Proof. Consider the subset $X' \times Y' = \{(x_1, y_1), (x_1, y_2), \dots, (x_1, y_n)\}$ of $X \times Y$ where $|X' \times Y'| = n$ which divides $|X \times Y| = n^2$. Suppose $g = (gx, gy) \in G$ such that $gx \in Stab_{S_n}(x_1)$. Then, g either fixes an element of $X' \times Y'$ or moves one element of $X' \times Y'$ to another so that $g(X' \times Y') = X' \times Y'$. Any other $g \in G$ takes an element of $X' \times Y'$ to an element not in $X' \times Y'$ so that $g(X' \times Y') \cap (X' \times Y') = \emptyset$. Thus, $X' \times Y'$ is a non-trivial block for the action. Therefore, the action imprimitive.

3. Rank and subdegrees of $S_n \times A_n$ on $X \times Y$

Lemma 3.1. The group $G = S_3 \times A_3$ acts on $X \times Y$ with rank 6 and subdegrees 1,1,1,2,2,2.

The Stabilizer for the action is $Stab_G(x_1, y_1) = \{(e_X, e_Y), ((x_2x_3), e_Y)\}$. The number of elements in $X \times Y$ fixed by the elements of H is given in the Table below.

Table 1: Elements of H and Corresponding Number of Fixed Points

Type of ordered pair (gX, gY) of permutation in H	Corresponding number of ordered pairs in H	$\lvert fix(gX, gY)\rvert = \lvert fix(gX)\rvert\lvert fix(gY)\rvert$ in $X \times Y$	*Total* ($col2 \times col3$)
(e_X, e_Y)	1	9	9
$((ab), e_Y)$	1	3	3
Total	2		12

By Theorem 1.1, the number of orbits of suborbits of G on $X \times Y$ is

$$\frac{1}{|H|}\sum_{(gX,gY)} |fix(gX, gY) = \frac{1}{2}[9+3] = \frac{12}{2} = 6$$

The six suborbits of G are

$$\Delta_0 = Orb_{(x_1,y_1)}(x_1, y_1) = \{(x_1, y_1)\},$$

$$\Delta_1 = Orb_{(x_1,y_1)}(x_1,y_2) = \{(x_1,y_2)\},$$

$$\Delta_2 = Orb_{(x_1,y_1)}(x_1,y_3) = \{(x_1,y_3)\},$$

$$\Delta_3 = Orb_{(x_1,y_1)}(x_2,y_1) = \{(x_2,y_1),(x_3,y_1)\},$$

$$\Delta_4 = Orb_{(x_1,y_1)}(x_2,y_2) = \{(x_2,y_2),(x_3,y_2)\},$$

$$\Delta_5 = Orb_{(x_1,y_1)}(x_2,y_3) = \{(x_2,y_3),(x_3,y_3)\}.$$

So, the action has rank 6 and subdegrees 1, 1, 1,2,2,2.

Theorem 3.1. The action of $G = S_n \times A_n$ on $X \times Y$ has rank 4 and subdegrees $1,(n-1),(n-1),(n-1)^2$ for all $n \geq 4$.

Proof. Let $H = Stab_G(x_1,y_1)$ be as defined in Theorem 2.1 above. Then the orbits of H on $X \times Y$ are

$$\Delta_0 = Orb_{G_{(x_1,y_1)}}(x_1,y_1) = \{(x_1,y_1)\}, \text{ where } |\Delta_0| = 1$$

$$\Delta_1 = Orb_{G_{(x_1,y_1)}}(x_1,y_2) = \{(x_1,y_2),(x_1,y_3),\ldots,(x_1,y_n)\}, \text{ where } |\Delta_1| = n-1$$

$$\Delta_2 = Orb_{G_{(x_1,y_1)}}(x_2,y_1) = \{(x_2,y_1),(x_3,y_1),\ldots,(x_n,y_1)\}, \text{ where } |\Delta_2| = n-1$$

$$\Delta_3 = Orb_{G_{(x_1,y_1)}}(x_2,y_2) = \{(x_2,y_2),(x_2,y_3),\ldots,(x_2,y_n),$$

$$(x_3,y_2),(x_3,y_3),\ldots,(x_3,y_n),$$

$$(x_4,y_2),(x_4,y_3),\ldots,(x_4,y_n),$$

$$(x_5,y_2),(x_5,y_3),\ldots,(x_5,y_n),$$

$$\ldots,(x_n,y_2),(x_n,y_3),\ldots,(x_n,y_n)\}, \text{ with } |\Delta_3| = (n-1)^2.$$

To prove that these are the only suborbits of G, it suffices to show that $P = \{\Delta_0,\Delta_1,\Delta_2,\Delta_3\}$ is a partition of $X \times Y$.

Clearly, $\Delta_i \neq \emptyset$ for each $i = 0,1,2,3$ and $\Delta_i \cap \Delta_j = \emptyset$ if $i \neq j$ $(i,j = 0,1,2,3)$.

Now,

$$\sum_{i=1}^{3} |\Delta_i| = 1 + 2(n-1) + (n-1)^2$$

$$= n^2$$

$$= |X \times Y|$$

and it follows that $\bigcup_{i=1}^{3} \Delta_i = X \times Y$. Thus, P is a partition of $X \times Y$.

4. Pairing of the suborbits of $G = S_n \times A_n$ on $X \times Y$

Theorem 4.1. The suborbits of $G = S_3 \times A_3$ on $X \times Y$ are self-paired except for a few.

Proof. The action of $G = S_3 \times A_3$ on $X \times Y$ has 5 non-trivial suborbits as $\Delta_1, \Delta_1, \Delta_3, \Delta_4$ and Δ_5. Since $|G|$ is even, then the action has at least one self-paired suborbit. Consider $(x_1, y_2) \in \Delta_1$ and $g = (e_X, (y_1 y_3 y_2)) \in G$. Then $g(x_1, y_2) = (x_1, y_1)$ and $g(x_1, y_1) = (x_1, y_3) \in \Delta_2$. So, $\Delta_1^* = \Delta_2$. Next, consider $(x_2, y_1) \in \Delta_3$ and also $g = ((x_1 x_2), e_Y) \in G$. Then $g(x_2, y_1) = (x_1, y_1)$ and therefore, $g(x_1, y_1) = (x_2, y_1) \in \Delta_3$. Finally, consider $(x_2, y_2) \in \Delta_4$. Suppose $g = ((x_1 x_2), (y_1 y_3 y_2)) \in G$. Then $g(x_2, y_2) = (x_1, y_1)$ and hence it is seen that $g(x_1, y_1) = (x_2, y_3) \in \Delta_5$. So, $\Delta_4^* = \Delta_5$.

Theorem 4.2. The suborbits of $G = S_n \times A_n$ on $X \times Y$ are self-paired for all $n \geq 4$.

Proof. From Theorem 3.2, then G has 3 non-trivial suborbits, namely Δ_1, Δ_2 and Δ_3. Consider $(x_1, y_2) \in \Delta_1$ and $g = (e_X, (y_1 y_3 y_2)) \in G$. Then we have that $g(x_1, y_2) = (x_1, y_1)$ and $g(x_1, y_1) = (x_1, y_3) \in \Delta_1$. So, $\Delta_1^* = \Delta_1$. Next, consider $(x_2, y_1) \in \Delta_2$ and $g = ((x_1 x_2), e_Y) \in G$. Then $g(x_2, y_1) = (x_1, y_1)$ and therefore, it can be seen that $g(x_1, y_1) = (x_2, y_1) \in \Delta_2$. So, $\Delta_2^* = \Delta_2$. Finally, consider $(x_2, y_2) \in \Delta_3$. Suppose $g = ((x_1 x_2), (y_1 y_3 y_2)) \in G$. Then $g(x_2, y_2) = (x_1, y_1)$ and therefore $g(x_1, y_1) = (x_2, y_3) \in \Delta_3$. So, $\Delta_3^* = \Delta_3$.

5. Suborbital graphs of $S_n \times A_n$ on $X \times Y$

A suborbital graph of the action has $X \times Y$ as its vertex set. Since for $n \geq 4$ all the suborbits are self-paired, then the corresponding suborbital graphs are undirected. Now, the construction and properties of the three non-trivial graphs of the action are as follows:

(i) The suborbital O_1 corresponding to the suborbit Δ_1 is

$$O_1 = \{((g_X, g_Y)(x_1, y_1), (g_X, g_Y)(x_1, y_2)) | (g_X, g_Y) \in G, (x_1, y_2) \in \Delta_1\}.$$

Thus, the suborbital graph Γ_1 corresponding to the suborbital O_1 has an edge from vertex (u, v) to vertex (x, y) if and only if $u = x$ and $v \neq y$. The graph is regular of degree $(n-1)$ since vertex (u, v) is connected to all the $(n-1)$ vertices (u, w) where $v \neq w$. It is disconnected since there is no path from vertex (u, v) to vertex (x, y) if $u \neq x$. Clearly, a connected component of the graph consists of n vertices so that there are $\frac{|X \times Y|}{n} = n$ connected components in the graph. It has girth 3 since $(x_1, y_1), (x_1, y_2)$ and (x_1, y_3) form cycle in Γ_1.

(ii) The suborbital O_2 corresponding to the suborbit Δ_2 is

$$O_2 = \{((g_X, g_Y)(x_1, y_1), (g_X, g_Y)(x_2, y_1)) | (g_X, g_Y) \in G, (x_2, y_1) \in \Delta_2\}.$$

The suborbital graph Γ_2 corresponding to the suborbital O_2 has an edge from vertex (u, v) to vertex (x, y) if and only if $u \neq x$ and $v = y$. This graph is isomorphic to Γ_1 and therefore, the two have the same properties.

(iii) The suborbital O_3 corresponding to the suborbit Δ_3 is

$$O_3 = \{((g_X, g_Y)(x_1, y_1), (g_X, g_Y)(x_2, y_2)) | (g_X, g_Y) \in G, (x_2, y_2) \in \Delta_3\}.$$

The suborbital graph Γ_3 corresponding to the suborbital O_3 has an edge from vertex (u, v) to vertex (x, y) if and only if $\{u, v\} \cap \{x, y\} = \emptyset$. The graph is regular of degree $(n-1)^2$ since vertex (u, v) is connected to all the $(n-1)^2$ vertices (x, y) where $\{u, v\} \cap \{x, y\} = \emptyset$. The graph is connected since there is a path between any two distinct vertices. It has girth 3 since the vertices $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) form cycle in Γ_3 since the vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are pairwise adjacent.

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